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Eigenvalues of matrices with several prescribed blocks

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Abstract

A previous paper by D. Hershkowitz [Linear and Multilinear Algebra 14 (1983) 315–342] described the possible eigenvalues of an $n \times n$ matrix when $2n - 3$ entries are fixed and the others vary. This paper describes the possible eigenvalues of a $pk \times pk$ matrix, partitioned into $k \times k$ blocks of size $p \times p$ when $2k - 3$ blocks are fixed and the others vary. © 2000 Published by Elsevier Science Inc. All rights reserved.

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1. Main results

Let F be an arbitrary field.

Hershkowitz [2] described the possible eigenvalues of an $n \times n$ matrix when $2n - 3$ entries are fixed and the others vary. For related results, see the references, for example.

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PII: S 0 0 2 4 - 3 7 9 5 (0 0) 0 0 0 3 9 - 2

Now let k, p be positive integers and $n = kp$. Let $(r_1,\ s_1), \ldots, (r_{2k-3},\ s_{2k-3}) \in \{1, \ldots, k\} \times \{1, \ldots, k\}$. Let $A_{r_i,\ s_i} \in F^{p\times p}$, $i \in \{1, \ldots, 2k-3\}$. Let $c_1, \ldots, c_n \in F$. The purpose of this paper is to solve the following problem and, therefore, to generalize Hershkowitz' theorem.

Problem. Under what conditions does there exist a matrix

$$\begin{bmatrix} C_{1,1} & \cdots & C_{1,k} \\ \vdots & & \vdots \\ C_{k,1} & \cdots & C_{k,k} \end{bmatrix}, \tag{1}$$

where the blocks $C_{i,j}$ are of size $p \times p$, with eigenvalues $c_1, \ldots, c_n$, such that $C_{r_i,s_i} = A_{r_i,s_i}$, $i \in \{1, \ldots, 2k-3\}$?

When $k = 2$, several results are known that study the possible eigenvalues of (1), when some blocks are fixed and the others vary. For example, see Lemmas 4, 7 and 8, which appeared in other papers.

For convenience of presentation, the solution of this problem is split through the next three theorems. The first one studies the case where all the blocks of one row or column are fixed. Without loss of generality, we assume that all the blocks of the first row are fixed. The second one studies the case where all the principal blocks are fixed and the third one studies the remaining cases.

As Hershkowitz [2] had already noticed, there are additional necessary conditions if more than $2k-3$ blocks are fixed. It is easy to find counterexamples. For example, given a matrix of the form (1), the interlacing inequalities for the invariant factors [7,11] may imply that some of the roots of the invariant factors of

$$\begin{bmatrix} -C_{1,2} & -C_{1,3} & \cdots & -C_{1,k} \\ xI_p - C_{2,2} & -C_{2,3} & \cdots & -C_{2,k} \end{bmatrix}$$

are eigenvalues of (1).

Theorem 1. *Suppose that all the blocks of the first row are prescribed. Let* $f_1(x) \mid \cdots \mid f_p(x)$ *be the invariant factors of*

$$\begin{bmatrix} xI_p - A_{1,1} & -A_{1,2} & \cdots & -A_{1,k} \end{bmatrix}. \tag{2}$$

There exists a matrix of the form (1), *where the blocks* $C_{i,j}$ *are of size* $p \times p$, *with eigenvalues* $c_1, \ldots, c_n$, *such that* $C_{r_i,\ s_i} = A_{r_i,\ s_i}$, $i \in \{1, \ldots, 2k-3\}$, *if and only if*

$$f_1 \cdots f_p \mid (x - c_1) \cdots (x - c_n). \tag{3}$$

Theorem 2. *Suppose that all the principal blocks are prescribed. Then there exists a matrix of the form* (1), *where the blocks* $C_{i,j}$ *are of size* $p \times p$, *with eigenvalues* $c_1, \ldots, c_n$, *such that* $C_{r_i,\ s_i} = A_{r_i,\ s_i}$, $i \in \{1, \ldots, 2k-3\}$, *if and only if*

$$\sum_{i=1}^{k} \operatorname{trace} A_{i,i} = \sum_{j=1}^{n} c_j. \tag{4}$$

Theorem 3. *Suppose that at least one principal block is free and at least one block in each row and each column is free. Then there exists a matrix of the form* (1), *where the blocks* $C_{i,j}$ *are of size* $p \times p$, *with eigenvalues* $c_1, \ldots, c_n$, *such that* $C_{r_i,\, s_i} = A_{r_i,\, s_i}$, $i \in \{1, \ldots, 2k-3\}$.

2. Proofs of the theorems

Throughout this section, ‘w.l.o.g’ means ‘without loss of generality’. Our proofs are split into many cases, because of the different positions where the prescribed blocks can be. In order to avoid a larger number of cases, we omit cases that can be reduced, w.l.o.g., to the studied ones using simple similarity transformations, like simultaneous permutations of rows and columns or transposition. ‘Subcase 2.1.3’ means ‘Subcase 3 of Subcase 1 of Case 2’.

Lemma 4 [12]. *Let* $B_{1,1} \in F^{p\times p}$, $B_{1,2} \in F^{p\times q}$, $c_1, \ldots, c_{p+q} \in F$. *Let* $f_1(x) | \cdots | f_p(x)$ *be the invariant factors of* $[xI_p - B_{1,1} \quad -B_{1,2}]$. *There exist* $B_{2,1} \in F^{q\times p}$ *and* $B_{2,2} \in F^{q\times q}$ *such that*

$$\begin{bmatrix} B_{1,1} & B_{1,2} \\ B_{2,1} & B_{2,2} \end{bmatrix} \tag{5}$$

has eigenvalues $c_1, \ldots, c_{p+q}$ *if and only if* $f_1 \cdots f_p \mid (x - c_1) \cdots (x - c_{p+q})$.

Lemma 5. *Let* $D_1 \in F^{p\times q}$, $D_2 \in F^{p\times r}$, *with* $q \geqslant p$. *Then there exists* $W \in F^{r\times q}$ *such that* $\operatorname{rank}(D_1 - D_2W) = \operatorname{rank}[D_1 \;\; D_2]$.

Proof. Let $d_1 = \operatorname{rank} D_1$, $d_2 = \operatorname{rank}[D_1 \;\; D_2] - d_1$. Let $Q \in F^{q\times q}$ and $R \in F^{r\times r}$ be nonsingular matrices such that the first d_1 and the last d_2 columns of $[D_1Q \;\; D_2R]$ are linearly independent. Then

$$\begin{aligned} q \geqslant p \geqslant \operatorname{rank}[D_1 \;\; D_2] &= d_1 + d_2 = \operatorname{rank}(D_1Q - D_2R(0_{r-d_2,q-d_2} \oplus I_{d_2})) \\ &= \operatorname{rank}(D_1 - D_2W), \end{aligned}$$

with $W = R(0_{r-d_2,q-d_2} \oplus I_{d_2})Q^{-1}$. □

Lemma 6. *Let* $D_0 \in F[x]^{p\times l}$, $D_1 \in F^{p\times q}$, $D_2 \in F^{p\times r}$, $W \in F^{r\times q}$. *Suppose that* $\operatorname{rank}(D_1 - D_2W) = \operatorname{rank}[D_1 \;\; D_2]$. *Then, for every* $Y \in F^{r\times l}$, *the matrices* $[D_0 \;\; D_1 \;\; D_2]$ *and* $[D_0 - D_2Y \;\; D_1 - D_2W]$ *have the same invariant factors.*

Proof. As $\operatorname{rank}(D_1 - D_2W) = \operatorname{rank}[D_1 \;\; D_2]$, the columns of D_2 are linear combinations of the columns of $D_1 - D_2W$, that is, $D_2 = (D_1 - D_2W)Z$, for some $Z \in F^{q\times r}$. Then

$$[D_0 \;\; D_1 \;\; D_2] = [D_0 - D_2Y \;\; D_1 - D_2W \;\; 0]\begin{bmatrix} I_l & 0 & 0 \\ 0 & I_q & Z \\ 0 & 0 & I_r \end{bmatrix}\begin{bmatrix} I_l & 0 & 0 \\ 0 & I_q & 0 \\ Y & W & I_r \end{bmatrix}.$$

Therefore, $[D_0 \;\; D_1 \;\; D_2]$ and $[D_0 - D_2Y \;\; D_1 - D_2W]$ have the same invariant factors. $\square$

Proof of Theorem 1. The necessity of (3) has been stated in Lemma 4.

Conversely, suppose that (3) is satisfied. The proof is by induction on k. Note that $k \geqslant 3$. If $k = 3$, then the fixed blocks are exactly the ones in the first row. This is a particular case of Lemma 4.

Now suppose that $k \geqslant 4$. Note that, there exists at least one row of blocks with all its blocks free. W.l.o.g., assume that the kth row of blocks is free. Let $C_{k,k} \in F^{p\times p}$ be a matrix with eigenvalues $c_{n-p+1}, \ldots, c_n$. Choose $W \in F^{p\times p}$ such that

$$\operatorname{rank}(A_{1,k-1} - A_{1,k}W) = \operatorname{rank}[A_{1,k-1} \quad A_{1,k}].$$

Case 1. Suppose that there are at least two prescribed blocks in the kth column. For every $i \in \{1, \ldots, k-1\}$, if the positions $(i, k-1)$ and (i, k) are both prescribed, let $A'_{i,k-1} = A_{i,k-1} - A_{i,k}W$. For every $i, j \in \{1, \ldots, k-1\}$, if (i, j) is prescribed and, either $j < k-1$ or (i, k) is free, let $A'_{i,j} = A_{i,j}$. According to Lemma 6, (2) and

$$\left[xI_p - A'_{1,1} \quad -A'_{1,2} \quad \cdots \quad -A'_{1,k-1}\right]. \tag{6}$$

have the same invariant factors. According to the induction assumption, there exists a matrix of the form

$$\begin{bmatrix} C_{1,1} & \cdots & C_{1,k-1} \\ \vdots & & \vdots \\ C_{k-1,1} & \cdots & C_{k-1,k-1} \end{bmatrix}, \tag{7}$$

with eigenvalues $c_1, \ldots, c_{n-p}$ such that $C_{r_i,\, s_i} = A'_{r_i,\, s_i}$ for every $i \in \{1, \ldots, 2k-3\}$ such that $r_i < k,\ s_i < k$. Let C be the matrix of the form

$$\begin{bmatrix} C_{1,1} & \cdots & C_{1,k-1} & C_{1,k} \\ \vdots & & \vdots & \vdots \\ C_{k-1,1} & \cdots & C_{k-1,k-1} & C_{k-1,k} \\ 0 & \cdots & 0 & C_{k,k} \end{bmatrix}, \tag{8}$$

where, for every $i \in \{1, \ldots, k-1\}$, $C_{i,k} = A_{i,k}$ if the position (i, k) is prescribed and $C_{i,k} = 0$ otherwise. The matrix C has eigenvalues $c_1, \ldots, c_n$ and is similar to $C' = X^{-1}CX$, where

$$X = I_{n-2p} \oplus \begin{bmatrix} I_p & 0 \\ W & I_p \end{bmatrix}.$$

The matrix C' has the prescribed form.

Case 2. Suppose that $(1, k)$ is the only prescribed position in the kth column of blocks. There exist $r, s, t \in \{1, \ldots, k-1\}$, with $r, t > 1$, such that (r, s) is prescribed and (r, t) is free. W.l.o.g., assume that $t = k-1$. Let $A'_{1,s} = A_{1,s} - A_{1,k}$, $A'_{1,k-1} = A_{1,k-1} - A_{1,k}W$. For every $i, j \in \{1, \ldots, k-1\}$, if (i, j) is prescribed and $(i, j) \notin \{(1, s), (1, k-1)\}$, let $A'_{i,j} = A_{i,j}$. According to Lemma 6, (2) and (6) have the same invariant factors. According to the induction assumption, there exists a matrix of the form (7) with eigenvalues $c_1, \ldots, c_{n-p}$ such that $C_{r_i,\, s_i} = A'_{r_i,\, s_i}$ for every $i \in \{1, \ldots, 2k-3\}$ such that $r_i < k,\ s_i < k,\ (r_i,\ s_i) \neq (r, s)$. Let C be the matrix of the form (8), where $C_{1,k} = A_{1,k},\ C_{r,k} = A_{r,s} - C_{r,s}$ and $C_{i,k} = 0$, $i \in \{2, \ldots, k-1\} \backslash \{r\}$. The matrix C has eigenvalues $c_1, \ldots, c_n$ and is similar to $C' = X^{-1}CX$, where

$$X = I_{(s-1)p} \oplus \begin{bmatrix} I_p & 0 & 0 & 0 \\ 0 & I_{(k-s-2)p} & 0 & 0 \\ 0 & 0 & I_p & 0 \\ I_p & 0 & W & I_p \end{bmatrix}.$$

The matrix C' has the prescribed form. $\square$

Lemma 7 [9]. *Let $B_{1,1} \in F^{p\times p}$, $B_{2,2} \in F^{q\times q}$, $c_1, \ldots, c_{p+q} \in F$. Suppose that $p \geqslant q$ and at least one of the matrices $B_{1,1}$, $B_{2,2}$ is nonscalar. Let $f_1(x)|\cdots|f_p(x)$ be the invariant factors of $xI_p - B_{1,1}$. Then there exist $B_{1,2} \in F^{p\times q}$ and $B_{2,1} \in F^{q\times p}$ such that* (5) *has eigenvalues $c_1, \ldots, c_{p+q}$ if and only if $f_1 \cdots f_{p-q}|(x-c_1)\cdots(x-c_{p+q})$ and* $\operatorname{trace}(B_{1,1} + B_{2,2}) = c_1 + \cdots + c_{p+q}$.

Proof of Theorem 2. The necessity is obvious. We shall prove the sufficiency by induction on k. Suppose that $k = 3$. Note that

$$\begin{bmatrix} xI_p - A_{1,1} & -I_p \\ 0 & xI_p - A_{2,2} \end{bmatrix}$$

has at least p invariant factors equal to 1. According to Lemma 7, there exist $C_{1,3}$, $C_{2,3},\ C_{3,1},\ C_{3,2} \in F^{p\times p}$ such that

$$\begin{bmatrix} A_{1,1} & I_p & C_{1,3} \\ 0 & A_{2,2} & C_{2,3} \\ C_{3,1} & C_{3,2} & A_{3,3} \end{bmatrix}$$

has eigenvalues $c_1, \ldots, c_n$. Now suppose that $k \geqslant 4$. Note that there exists a row of blocks with exactly one prescribed position. W.l.o.g., suppose that (k, k) is the only prescribed position in the kth row. Let $C_{k,k}$ be a matrix with eigenvalues $c_{n-p+1}, \ldots, c_n$.

Case 1. Suppose that there exists at least one prescribed nonprincipal position in the kth column of blocks. Note that there exists at least one free position in the kth column of blocks. W.l.o.g., suppose that $(1, k)$ is free. Let $A'_{1,1} = A_{1,1} +$

$A_{k,k} - C_{k,k}$. For every $i \in \{2, \ldots, k-1\}$, if the positions $(i, 1)$ and (i, k) are both prescribed, let $A'_{i,1} = A_{i,1} - A_{i,k}$. For every $i, j \in \{1, \ldots, k-1\}$, if (i, j) is prescribed, $(i, j) \neq (1, 1)$, and, either $j > 1$ or (i, k) is free, let $A'_{i,j} = A_{i,j}$. According to the induction assumption, there exists a matrix of the form (7) with eigenvalues $c_1, \ldots, c_{n-p}$ such that $C_{r_i,s_i} = A'_{r_i,s_i}$, for every $i \in \{1, \ldots, 2k-3\}$ such that $r_i < k, s_i < k$. Let C be the matrix of the form (8), where $C_{1,k} = C_{k,k} - A_{k,k}$, $C_{i,k} = A_{i,k}$ if (i, k) is prescribed and $C_{i,k} = 0$ if (i, k) is free, $i \in \{2, \ldots, k-1\}$. The matrix C has eigenvalues $c_1, \ldots, c_n$ and is similar to $C' = X^{-1}CX$, where

$$X = \begin{bmatrix} I_p & 0 & 0 \\ 0 & I_{n-2p} & 0 \\ I_p & 0 & I_p \end{bmatrix}. \tag{9}$$

The matrix C' has the prescribed form.

Case 2. Suppose that there are no prescribed nonprincipal positions in the kth column of blocks. Then there exists a nonprincipal prescribed position (r, s), with $1 \leqslant r, s < k$. W.l.o.g., suppose that $(r, s) = (2, 1)$. Let $A'_{1,1} = A_{1,1} + A_{k,k} - C_{k,k}$. For every $i, j \in \{1, \ldots, k-1\}$, if (i, j) is prescribed and $(i, j) \neq (1, 1)$, let $A'_{i,j} = A_{i,j}$. According to the induction assumption, there exists a matrix of the form (7) with eigenvalues $c_1, \ldots, c_{n-p}$ such that $C_{r_i,s_i} = A'_{r_i,\, s_i}$, for every $i \in \{1, \ldots, 2k-3\}$ such that $r_i < k,\ s_i < k,\ (r_i,\ s_i) \neq (2, 1)$. Let C be the matrix of the form (8), where $C_{1,k} = C_{k,k} - A_{k,k},\ C_{2,k} = A_{2,1} - C_{2,1},\ C_{i,k} = 0,\ i \in \{3, \ldots, k-1\}$. The matrix C has eigenvalues $c_1, \ldots, c_n$ and is similar to $C' = X^{-1}CX$, where X has the form (9). The matrix C' has the prescribed form. □

Lemma 8 [5]. *Let $B_{1,1} \in F^{p\times p}$, $c_1, \ldots, c_{p+q} \in F$. Let $f_1(x) \mid \cdots \mid f_p(x)$ be the invariant factors of $xI_p - B_{1,1}$. Then there exist $B_{1,2} \in F^{p\times q}$, $B_{2,1} \in F^{q\times p}$ and $B_{2,2} \in F^{q\times q}$ such that* (5) *has eigenvalues $c_1, \ldots, c_{p+q}$ if and only if $f_1 \cdots f_{p-q} \mid (x - c_1)\cdots(x - c_{p+q})$.* (*Make convention that $f_1 \cdots f_{p-q} = 1$ if $p - q < 1$.*)

Proof of Theorem 3. By induction on k. Suppose that $k = 2$. Then there is exactly one prescribed position. If a principal position is prescribed, the result follows from Lemma 8. If a nonprincipal position is prescribed, the result is trivial.

From now on, suppose that $k \geqslant 3$. Let $C_{k,k} \in F^{p\times p}$ be a matrix with eigenvalues $c_{n-p+1}, \ldots, c_n$.

Case 1. Suppose that there exists a row or column of blocks with all the positions free. W.l.o.g., suppose that the kth row has all its positions free.

Subcase 1.1. Suppose that, for every $i \in \{1, \ldots, k-1\}$, at least one of the positions $(i, 1), \ldots, (i, k-1)$ is free, at least one of the positions $(1, i), \ldots, (k-1, i)$ is free, and at least one of the positions $(1, 1), \ldots, (k-1, k-1)$ is free.

Subcase 1.1.1. Suppose that at least two positions in the kth column are prescribed. According to the induction assumption, there exists a matrix of the form (7) with eigenvalues $c_1, \ldots, c_{n-p}$ such that $C_{r_i,s_i} = A_{r_i,s_i}$, for every $i \in \{1, \ldots, 2k-3\}$ such that $r_i < k,\ s_i < k$. Let C be the matrix of the form (8), where $C_{i,k} = A_{i,k}$

if (i,k) is prescribed and $C_{i,k}=0$ if (i,k) is free, $i\in\{1,\dots,k-1\}$. The matrix C has eigenvalues $c_1,\dots,c_n$.

Subcase 1.1.2. Suppose that exactly one position in the *k*th column is prescribed. W.l.o.g., assume that $(1,k)$ is prescribed. There exists a prescribed position (r,s), with $1<r<k,\ 1\leqslant s<k$. If $(1,s)$ is prescribed, let $A'_{1,s}=A_{1,s}-A_{1,k}$. For every $i,j\in\{1,\dots,k-1\}$, if (i,j) is prescribed and $(i,j)\neq(1,s)$, let $A'_{i,j}=A_{i,j}$. According to the induction assumption, there exists a matrix of the form (7) with eigenvalues $c_1,\dots,c_{n-p}$ such that $C_{r_i,\,s_i}=A'_{r_i,\,s_i}$, for every $i\in\{1,\dots,2k-3\}$ such that $r_i<k,\ s_i<k,\ (r_i,\ s_i)\neq(r,s)$. Let C be the matrix of the form (8), where $C_{1,k}=A_{1,k}$, $C_{r,k}=A_{r,s}-C_{r,s}$ and $C_{i,k}=0$ if $i\in\{2,\dots,k-1\}\backslash\{r\}$. The matrix C has eigenvalues $c_1,\dots,c_n$ and is similar to $C'=X^{-1}CX$, where

$$X=I_{(s-1)p}\oplus\begin{bmatrix} I_p & 0 & 0\\ 0 & I_{(k-s-1)p} & 0\\ I_p & 0 & I_p\end{bmatrix}. \tag{10}$$

The matrix C' has the prescribed form.

Subcase 1.1.3. Suppose that all the positions in the *k*th column are free. There exist two prescribed positions (r,s) and (r',s), with $1\leqslant r<r'<k$, $1\leqslant s<k$. According to the induction assumption, there exists a matrix of the form (7) with eigenvalues $c_1,\dots,c_{n-p}$ such that $C_{r_i,\,s_i}=A_{r_i,\,s_i}$ for every $i\in\{1,\dots,2k-3\}$ such that $r_i<k,\ s_i<k,\ (r_i,\ s_i)\notin\{(r,s),(r',s)\}$. Let C be the matrix of the form (8), where $C_{r,k}=A_{r,s}-C_{r,s}$, $C_{r',k}=A_{r',s}-C_{r',s}$ and $C_{i,k}=0$ if $i\in\{1,\dots,k-1\}\backslash\{r,r'\}$. The matrix C has eigenvalues $c_1,\dots,c_n$ and is similar to $C'=X^{-1}CX$, where X has the form (10). The matrix C' has the prescribed form.

Subcase 1.2. Suppose that there exists $i\in\{1,\dots,k-1\}$ such that all the positions $(i,1),\dots,(i,k-1)$ are prescribed. W.l.o.g., assume that $(1,1),\dots,(1,k-1)$ are prescribed.

Subcase 1.2.1. Suppose that at least one position in the *k*th column is prescribed. There exists $s\in\{1,\dots,k-1\}$ such that $(1,s)$ is the only position prescribed in the *s*th column. According to the induction assumption, there exists a matrix of the form (7) with eigenvalues $c_1,\dots,c_{n-p}$ such that $C_{r_i,\,s_i}=A_{r_i,\,s_i}$, for every $i\in\{1,\dots,2k-3\}$ such that $r_i<k,\ s_i<k,\ (r_i,\ s_i)\neq(1,s)$. Let C be the matrix of the form (8), where $C_{1,k}=A_{1,s}-C_{1,s}$, $C_{i,k}=A_{i,k}$ if (i,k) is prescribed and $C_{i,k}=0$ if (i,k) is free, $i\in\{2,\dots,k-1\}$. The matrix C has eigenvalues $c_1,\dots,c_n$ and is similar to $C'=X^{-1}CX$, where X has the form (10). The matrix C' has the prescribed form.

Subcase 1.2.2. Suppose that all the positions in the *k*th column are free. According to the induction assumption, there exists a matrix of the form (7) with eigenvalues $c_1,\dots,c_{n-p}$ such that $C_{r_i,\,s_i}=A_{r_i,\,s_i}$, for every $i\in\{1,\dots,2k-3\}$ such that $r_i<k,\ s_i<k,\ (r_i,\ s_i)\notin\{(1,1),(1,2)\}$. Let C be the matrix of the form (8), where $C_{1,k}=I_p$ and $C_{i,k}=0$, $i\in\{2,\dots,k-1\}$. The matrix C has eigenvalues $c_1,\dots,c_n$ and is similar to $C'=X^{-1}CX$, where

$$X = \begin{bmatrix} I_p & 0 & 0 & 0 \\ 0 & I_p & 0 & 0 \\ 0 & 0 & I_{n-3p} & 0 \\ A_{1,1} - C_{1,1} & A_{1,2} - C_{1,2} & 0 & I_p \end{bmatrix}.$$

The matrix C' has the prescribed form.

Subcase 1.3. Suppose that there exists $i \in \{1, \dots, k-1\}$ such that all the positions $(1, i), \dots, (k-1, i)$ are prescribed. W.l.o.g., assume that $(1, 1), \dots, (k-1, 1)$ are prescribed.

Subcase 1.3.1. Suppose that at least one position in the *k*th column is prescribed. There exists $r \in \{1, \dots, k-1\}$ such that (r, k) is free. For each $i \in \{1, \dots, k-1\}$, if (i, k) is prescribed, let $A'_{i,1} = A_{i,1} - A_{i,k}$. For each $i, j \in \{1, \dots, k-1\}$, if (i, j) is prescribed and, either $j > 1$ or (i, k) is free, let $A'_{i,j} = A_{i,j}$. According to the induction assumption, there exists a matrix of the form (7) with eigenvalues $c_1, \dots, c_{n-p}$ such that $C_{r_i, s_i} = A'_{r_i, s_i}$, for every $i \in \{1, \dots, 2k-3\}$ such that $r_i < k,\ s_i < k,\ (r_i,\ s_i) \neq (r, 1)$. Let C be the matrix of the form (8), where $C_{r,k} = A_{r,1} - C_{r,1}$, $C_{i,k} = A_{i,k}$ if (i, k) is prescribed and $C_{i,k} = 0$ if (i, k) is free, $i \in \{1, \dots, k-1\} \backslash \{r\}$. The matrix C has eigenvalues $c_1, \dots, c_n$ and is similar to $C' = X^{-1}CX$, where where X has the form (9). The matrix C' has the prescribed form.

Subcase 1.3.2. Suppose that all the positions in the *k*th column are free. According to the induction assumption, there exists a matrix of the form (7) with eigenvalues $c_1, \dots, c_{n-p}$ such that $C_{r_i, s_i} = A_{r_i, s_i}$, for every $i \in \{1, \dots, 2k-3\}$ such that $r_i < k,\ s_i < k,\ (r_i,\ s_i) \notin \{(1, 1), (2, 1)\}$. Let C be the matrix of the form (8), where $C_{1,k} = A_{1,1} - C_{1,1},\ C_{2,k} = A_{2,1} - C_{2,1}$ and $C_{i,k} = 0,\ i \in \{3, \dots, k-1\}$. The matrix C has eigenvalues $c_1, \dots, c_n$ and is similar to $C' = X^{-1}CX$, where X has the form (9). The matrix C' has the prescribed form.

Subcase 1.4. Suppose that, for every $i \in \{1, \dots, k-1\}$, at least one of the positions $(i, 1), \dots, (i, k-1)$ is free, at least one of the positions $(1, i), \dots, (k-1, i)$ is free, and that all the positions $(1, 1), \dots, (k-1, k-1)$ are prescribed.

Subcase 1.4.1. Suppose that at least one position in the *k*th column is prescribed. There exists $r \in \{1, \dots, k-1\}$ such that (r, k) is free. W.l.o.g., assume that $r = 1$. For each $i \in \{2, \dots, k-1\}$, if $(i, 1)$ and (i, k) are both prescribed, let $A'_{i,1} = A_{i,1} - A_{i,k}$. For each $i, j \in \{1, \dots, k-1\}$, if (i, j) is prescribed and, either $j > 1$ or (i, k) is free, let $A'_{i,j} = A_{i,j}$. According to the induction assumption, there exists a matrix of the form (7) with eigenvalues $c_1, \dots, c_{n-p}$ such that $C_{r_i, s_i} = A'_{r_i, s_i}$, for every $i \in \{1, \dots, 2k-3\}$ such that $r_i < k,\ s_i < k,\ (r_i,\ s_i) \neq (1, 1)$. Let C be the matrix of the form (8), where $C_{1,k} = A_{1,1} - C_{1,1},\ C_{i,k} = A_{i,k}$ if (i, k) is prescribed and $C_{i,k} = 0$ if (i, k) is free, $i \in \{2, \dots, k-1\}$. The matrix C has eigenvalues $c_1, \dots, c_n$ and is similar to $C' = X^{-1}CX$, where X has the form (9). The matrix C' has the prescribed form.

Subcase 1.4.2. Suppose that all the positions in the *k*th column are free. There exists a prescribed nonprincipal position (r, s), with $r, s \in \{1, \dots, k-1\}$. W.l.o.g., assume that $(r, s) = (2, 1)$. According to the induction assumption, there exists a matrix of the form (7) with eigenvalues $c_1, \ldots, c_{n-p}$ such that $C_{r_i, s_i} = A_{r_i, s_i}$, for every $i \in \{1, \ldots, 2k-3\}$ such that $r_i < k,\ s_i < k,\ (r_i,\ s_i) \notin \{(1,1), (2,1)\}$. Let C be the matrix of the form (8), where $C_{1,k} = A_{1,1} - C_{1,1},\ C_{2,k} = A_{2,1} - C_{2,1}$ and $C_{i,k} = 0,\ i \in \{3, \ldots, k-1\}$. The matrix C has eigenvalues $c_1, \ldots, c_n$ and is similar to $C' = X^{-1}CX$, where X has the form (9). The matrix C' has the prescribed form.

Case 2. Suppose that there is no row or column of blocks with all the positions free. Then there exists a row with exactly one prescribed position. W.l.o.g., suppose that the kth row has exactly one prescribed position. Note that, for every $i \in \{1, \ldots, k-1\}$, at least one of the positions $(i, 1), \ldots, (i, k-1)$ is free and that at least one of the positions $(1, i), \ldots, (k-1, i)$ is free. (Otherwise there would be a row or a column with all the positions free.)

Subcase 2.1. Suppose that the position (k, k) is prescribed. Note that at least one of the positions $(1, 1), \ldots, (k-1,\ k-1)$ is free.

Subcase 2.1.1. Suppose that at least one nonprincipal position in the kth column is prescribed. There exists $i \in \{1, \ldots, k-1\}$ such that (i, k) is free. W.l.o.g., assume that $(1, k)$ is free. If $(1, 1)$ is prescribed, let $A'_{1,1} = A_{1,1} - C_{k,k} + A_{k,k}$. For every $i \in \{2, \ldots, k-1\}$, if $(i, 1)$ and (i, k) are prescribed, let $A'_{i,1} = A_{i,1} - A_{i,k}$. For every $i, j \in \{1, \ldots, k-1\}$, if (i, j) is prescribed, $(i, j) \neq (1, 1)$ and, either $j > 1$ or (i, k) is free, let $A'_{i,j} = A_{i,j}$. According to the induction assumption, there exists a matrix of the form (7) with eigenvalues $c_1, \ldots, c_{n-p}$ such that $C_{r_i, s_i} = A'_{r_i, s_i}$, for every $i \in \{1, \ldots, 2k-3\}$ such that $r_i < k,\ s_i < k$. Let C be the matrix of the form (8), where $C_{1,k} = C_{k,k} - A_{k,k},\ C_{i,k} = A_{i,k}$ if (i, k) is prescribed and $C_{i,k} = 0$ if (i, k) is free, $i \in \{2, \ldots, k-1\}$. The matrix C has eigenvalues $c_1, \ldots, c_n$ and is similar to $C' = X^{-1}CX$, where X has the form (9). The matrix C' has the prescribed form.

Subcase 2.1.2. Suppose that all the nonprincipal positions in the kth column are free. As there exists a free principal position, assume, w.l.o.g., that $(1, 1)$ is free. As the first column has at least a prescribed position, assume, w.l.o.g., that $(2, 1)$ is prescribed. According to the induction assumption, there exists a matrix of the form (7) with eigenvalues $c_1, \ldots, c_{n-p}$ such that $C_{r_i, s_i} = A_{r_i, s_i}$, for every $i \in \{1, \ldots, 2k-3\}$ such that $r_i < k,\ s_i < k,\ (r_i,\ s_i) \neq (2, 1)$. Let C be the matrix of the form (8), where $C_{1,k} = C_{k,k} - A_{k,k},\ C_{2,k} = A_{2,1} - C_{2,1},\ C_{i,k} = 0,\ i \in \{3, \ldots, k-1\}$. The matrix C has eigenvalues $c_1, \ldots, c_n$ and is similar to $C' = X^{-1}CX$, where X has the form (9). The matrix C' has the prescribed form.

Subcase 2.2. Suppose that the position (k, k) is free. W.l.o.g., assume that $(k, 1)$ is prescribed. Note that, if all the positions $(1, 1), \ldots, (k-1, k-1)$ are prescribed, then there exists $i \in \{1, \ldots, k-1\}$ such that (i, i) is the only prescribed position in the ith row. As this situation can be reduced to Subcase 2.1 by simultaneous permutations of rows and columns, we assume that at least one of the positions $(1, 1), \ldots, (k-1, k-1)$ is free.

Subcase 2.2.1. Suppose that the kth column has at least two prescribed nonprincipal positions. Note that there are at least three free positions in the first column. (Otherwise, there would be a column without prescribed positions.) W.l.o.g., assume that

$(k-1,1)$ is free. For each $i \in \{1,\ldots,k-1\}$, if $(i,k-1)$ and (i,k) are prescribed, let $A'_{i,k-1} = A_{i,k-1} - A_{i,k}$. For each $i, j \in \{1,\ldots,k-1\}$, if (i,j) is prescribed and, either $j < k-1$ or (i,k) is free, let $A'_{i,j} = A_{i,j}$. According to the induction assumption, there exists a matrix of the form (7) with eigenvalues $c_1,\ldots,c_{n-p}$ such that $C_{k-1,1} = -A_{k,1}$ and $C_{r_i,\, s_i} = A'_{r_i,\, s_i}$, for every $i \in \{1,\ldots,2k-3\}$ such that $r_i < k,\ s_i < k$. Let C be the matrix of the form (8), where $C_{i,k} = A_{i,k}$ if (i,k) is prescribed and $C_{i,k} = 0$ if (i,k) is free, $i \in \{1,\ldots,k-1\}$. The matrix C has eigenvalues $c_1,\ldots,c_n$ and is similar to $C' = X^{-1}CX$, where

$$X = I_{n-2p} \oplus \begin{bmatrix} I_p & 0 \\ I_p & I_p \end{bmatrix}.$$

The matrix C' has the prescribed form.

Subcase 2.2.2. Suppose that the kth column has exactly one prescribed nonprincipal position.

Subcase 2.2.2.1. Suppose that $(1,k)$ is free. W.l.o.g., assume that $(k-1,k)$ is prescribed.

Subcase 2.2.2.1.1. Suppose that $(1,1)$ is free. If $(k-1,1)$ is prescribed, let $A'_{k-1,1} = A_{k-1,1} - A_{k-1,k}$. For each $i, j \in \{1,\ldots,k-1\}$, if (i,j) is prescribed and $(i,j) \neq (k-1,1)$, let $A'_{i,j} = A_{i,j}$. According to the induction assumption, there exists a matrix of the form (7) with eigenvalues $c_1,\ldots,c_{n-p}$ such that $C_{r_i,\, s_i} = A'_{r_i,\, s_i}$, for every $i \in \{1,\ldots,2k-3\}$ such that $r_i < k,\ s_i < k$. Let C be the matrix of the form (8), where $C_{1,k} = C_{k,k} - C_{1,1} - A_{k,1}$, $C_{k-1,k} = A_{k-1,k}$, $C_{i,k} = 0,\ i \in \{2,\ldots,k-2\}$. The matrix C has eigenvalues $c_1,\ldots,c_n$ and is similar to $C' = X^{-1}CX$, where X has the form (9). The matrix C' has the prescribed form.

Subcase 2.2.2.1.2. Suppose that $(k-1,k-1)$ is free. Note that this subcase reduces to the previous one by transposition and simultaneous permutations of rows and columns.

Subcase 2.2.2.1.3. Suppose that $(1,1)$ and $(k-1,k-1)$ are prescribed.

Subcase 2.2.2.1.3.1. Suppose that, for every $i \in \{2,\ldots,k-2\}$, at least one of the entries $(i,1)$ and (i,i) is prescribed. Then there exists $j \in \{2,\ldots,k-2\}$ such that the jth column of block does not have a prescribed nonprincipal position. This situation has already been considered.

Subcase 2.2.2.1.3.2. Suppose that there exists $i \in \{2,\ldots,k-2\}$ such that $(i,1)$ and (i,i) are free. W.l.o.g., suppose that $(2,1)$ and $(2,2)$ are free. If the position $(k-1,1)$ is prescribed, let $A'_{k-1,1} = A_{k-1,1} - A_{k-1,k}$. If the position $(k-1,2)$ is prescribed, let $A'_{k-1,2} = A_{k-1,2} - A_{k-1,k}$. For every $i, j \in \{1,\ldots,k-1\}$, if (i,j) is prescribed and $(i,j) \notin \{(k-1,1),(k-1,2)\}$, let $A'_{i,j} = A_{i,j}$. According to the induction assumption, there exists a matrix of the form (7) with eigenvalues $c_1,\ldots,c_{n-p}$ such that $C_{r_i,\, s_i} = A'_{r_i,\, s_i}$, for every $i \in \{1,\ldots,2k-3\}$ such that $r_i < k,\ s_i < k$. Let C be the matrix of the form (8), where $C_{k-1,k} = A_{k-1,k},\ C_{2,k} = C_{k,k} - A_{1,1} - A_{k,1} - C_{2,1}$ and $C_{i,k} = 0$ if $i \in \{1,\ldots,k-2\}\backslash\{2\}$. The matrix C has eigenvalues $c_1,\ldots,c_n$ and is similar to $C' = X^{-1}CX$, where

$$X = \begin{bmatrix} I_p & 0 & 0 & 0 \\ 0 & I_p & 0 & 0 \\ 0 & 0 & I_{n-3p} & 0 \\ I_p & I_p & 0 & I_p \end{bmatrix}.$$

The matrix C' has the prescribed form.

Subcase 2.2.2.2. Suppose that $(1, k)$ is prescribed. As each row has at least one prescribed position, there exists $i \in \{2, \ldots, k-1\}$ such that the ith row has exactly one prescribed position. W.l.o.g., assume that the $(k-1)$th row has exactly one prescribed position. The situation where $(k-1, k-1)$ is prescribed can be reduced to Subcase 2.1. Therefore, assume that $(k-1, k-1)$ is free.

Subcase 2.2.2.2.1. Suppose that $(k-1, 1)$ is prescribed. Also assume that $(1, k-1)$ is the only position prescribed in the $(k-1)$th column, as all the other situations can be reduced to previously studied subcases. Choose a nonsingular matrix $W \in F^{2p\times 2p}$ such that

$$W \begin{bmatrix} A_{k-1,1} \\ A_{k,1} \end{bmatrix} = \begin{bmatrix} A'_{k-1,1} \\ 0 \end{bmatrix},$$

where $A'_{k-1,1} \in F^{p\times p}$. Let

$$\begin{bmatrix} A'_{1,k-1} & A'_{1,k} \end{bmatrix} := \begin{bmatrix} A_{1,k-1} & A_{1,k} \end{bmatrix} W^{-1},$$

where $A'_{1,k-1}, A'_{1,k} \in F^{p\times p}$. For each $i, j \in \{1, \ldots, k-1\}$, if (i, j) is prescribed and $(i, j) \notin \{(k-1, 1), (1, k-1)\}$, let $A'_{i,j} = A_{i,j}$. According to the induction assumption, there exists a matrix of the form (7) with eigenvalues $c_1, \ldots, c_{n-p}$ such that $C_{r_i,\, s_i} = A'_{r_i,\, s_i}$, for every $i \in \{1, \ldots, 2k-3\}$ such that $r_i < k,\ \ s_i < k$. Let C be the matrix of the form (8), where $C_{1,k} = A'_{1,k},\ C_{i,k} = 0,\ i \in \{2, \ldots, k-1\}$. The matrix C has eigenvalues $c_1, \ldots, c_n$ and is similar to $C' = X^{-1}CX$, where $X = I_{n-2p} \oplus W$. The matrix C' has the prescribed form.

Subcase 2.2.2.2.2. Suppose that $(k-1, 1)$ is free. W.l.o.g., assume that $(k-1, 2)$ is prescribed. Note that $k \geqslant 4$. Also assume that $(2, k-1)$ is the only position prescribed in the $(k-1)$th column, as all the other situations can be reduced to previously studied subcases. If $(1, 1)$ is prescribed, let $A'_{1,1} = A_{1,1} - A_{1,k}$. For each $i, j \in \{1, \ldots, k-1\}$, if (i, j) is prescribed and $(i, j) \neq (1, 1)$, let $A'_{i,j} = A_{i,j}$. According to the induction assumption, there exists a matrix of the form (7) with eigenvalues $c_1, \ldots, c_{n-p}$ such that $C_{r_i,\, s_i} = A'_{r_i,\, s_i}$, for every $i \in \{1, \ldots, 2k-3\}$ such that $r_i < k,\ \ s_i < k$. Let C be the matrix of the form (8), where $C_{1,k} = A_{1,k}$, $C_{k-1,k} = C_{k,k} - C_{1,1} - A_{1,k} - C_{k-1,1} - A_{k,1}$, $C_{i,k} = 0$, $i \in \{2, \ldots, k-2\}$. The matrix C has eigenvalues $c_1, \ldots, c_n$ and is similar to $C' = X^{-1}CX$, where

$$X = \begin{bmatrix} I_p & 0 & 0 & 0 \\ 0 & I_{n-3p} & 0 & 0 \\ 0 & 0 & I_p & 0 \\ I_p & 0 & I_p & I_p \end{bmatrix}.$$

The matrix C' has the prescribed form. $\square$

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